

Interpretation of Fluctuation Spectra in Lipid
Bilayer Simulations
Supporting Material

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S1. Derivation of the Fluctuation Spectra from the Molecular Positions

The treatment is similar to that of simple fluids (1). In the bilayer, complications rise due to the coupling of the two monolayers and fluctuations of the lipid/water interface. For each monolayer surface, we define the local height and local number density

$$z(\mathbf{r}) = a \sum_{l=1}^N z(\mathbf{r}_l) \delta(\mathbf{r} - \mathbf{r}_l) \quad (\text{S1.1})$$

$$\rho(\mathbf{r}) = \sum_{l=1}^N \delta(\mathbf{r} - \mathbf{r}_l) \quad , \quad (\text{S1.2})$$

from the positions $\mathbf{r}_l = (x_l, y_l)$ of the N constituent lipid molecules. The Fourier transforms are

$$z(\mathbf{q}) = \frac{1}{N} \sum_{l=1}^N z(\mathbf{r}_l) e^{-i\mathbf{q}\cdot\mathbf{r}_l} \quad (\text{S1.3})$$

$$\rho(\mathbf{q}) = \frac{1}{A} \sum_{l=1}^N e^{-i\mathbf{q}\cdot\mathbf{r}_l} \quad , \quad (\text{S1.4})$$

having the same dimensions as the real space functions. Following the notation of the monolayer functions of the main text (2), we define the bilayer undulations $u(\mathbf{r})$, thickness $h(\mathbf{r})$ and number density $\rho(\mathbf{r})$ from Eqs. (6), (7) and (8), and their concomitant Fourier representations according to Eqs. (10), (11) and (12); where all equations refer to Brandt et al. (2).

For a bilayer centered around $z = 0$, we have the spatial averages $\overline{u(\mathbf{r})} = u = 0$, $\overline{h(\mathbf{r})} = h$ (half the membrane thickness) and $\overline{\rho(\mathbf{r})} = \rho = a^{-1}$. We now wish to calculate the static structure factors $\langle |u(\mathbf{q})|^2 \rangle = \langle u(\mathbf{q}) u^*(\mathbf{q}) \rangle$, $\langle |h(\mathbf{q})|^2 \rangle = \langle h(\mathbf{q}) h^*(\mathbf{q}) \rangle$ and $\langle |\rho(\mathbf{q})|^2 \rangle = \langle \rho(\mathbf{q}) \rho^*(\mathbf{q}) \rangle$. Doing that, we get monolayer self terms such as $\langle z_1(\mathbf{q}) z_1^*(\mathbf{q}) \rangle$; and cross terms such as $\langle z_1(\mathbf{q}) z_2^*(\mathbf{q}) \rangle$. We temporarily ignore a δ -function contribution at $q = 0$. From Eq. (S1.3) for the heights and Eq. (S1.4) for the number densities, we

obtain the self terms with the diagonal $l = m$ terms separate,

$$\begin{aligned} \langle z_1(\mathbf{q}) z_1^*(\mathbf{q}) \rangle &= \frac{1}{N^2} \left\langle \sum_{l=1}^N z_1^2(\mathbf{r}_{1l}) \right\rangle \\ &\quad + \frac{1}{N^2} \left\langle \sum_{l \neq m}^N z_1(\mathbf{r}_{1l}) z_1(\mathbf{r}_{1m}) e^{-i\mathbf{q} \cdot (\mathbf{r}_{1l} - \mathbf{r}_{1m})} \right\rangle \end{aligned} \quad (\text{S1.5})$$

$$\langle \rho_1(\mathbf{q}) \rho_1^*(\mathbf{q}) \rangle = \frac{1}{a^2 N} + \frac{1}{(aN)^2} \left\langle \sum_{l \neq m}^N e^{-i\mathbf{q} \cdot (\mathbf{r}_{1l} - \mathbf{r}_{1m})} \right\rangle, \quad (\text{S1.6})$$

and similar results for layer 2. The first term in Eq. (S1.5) is

$$N^{-2} \left\langle \sum_{l=1}^N z_1^2(\mathbf{r}_{1l}) \right\rangle = N^{-1} \left\langle \overline{z_1^2(\mathbf{r}_1)} \right\rangle = N^{-1} \langle z_1^2 \rangle.$$

The second terms are obtained after integrating over total area and using δ -functions to pick out the molecular positions,

$$\begin{aligned} \langle z_1(\mathbf{q}) z_1^*(\mathbf{q}) \rangle &= \frac{\langle z_1^2 \rangle}{N} + \frac{1}{N^2} \iint_{A^2} d^2\mathbf{r} d^2\mathbf{r}' e^{-i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \\ &\quad \times \left\langle \sum_{l \neq m}^N z_1(\mathbf{r}) z_1(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}_{1l}) \delta(\mathbf{r}' - \mathbf{r}_{1m}) \right\rangle, \end{aligned} \quad (\text{S1.7})$$

$$\begin{aligned} \langle \rho_1(\mathbf{q}) \rho_1^*(\mathbf{q}) \rangle &= \frac{1}{a^2 N} + \frac{1}{(aN)^2} \iint_{A^2} d^2\mathbf{r} d^2\mathbf{r}' e^{-i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \\ &\quad \times \left\langle \sum_{l \neq m}^N \delta(\mathbf{r} - \mathbf{r}_{1l}) \delta(\mathbf{r}' - \mathbf{r}_{1m}) \right\rangle. \end{aligned} \quad (\text{S1.8})$$

The term enclosed in brackets are similar to the radial distribution function of the bulk liquid (1). The usual correlation functions are defined as

$$g_z(r = |\mathbf{r}' - \mathbf{r}''|) = \frac{a^2}{\langle h \rangle^2} \left\langle \sum_{l \neq m}^N z(\mathbf{r}') z(\mathbf{r}'') \delta(\mathbf{r}' - \mathbf{r}_l) \delta(\mathbf{r}'' - \mathbf{r}_m) \right\rangle \quad (\text{S1.9})$$

$$g(r = |\mathbf{r}' - \mathbf{r}''|) = a^2 \left\langle \sum_{l \neq m}^N \delta(\mathbf{r}' - \mathbf{r}_l) \delta(\mathbf{r}'' - \mathbf{r}_m) \right\rangle, \quad (\text{S1.10})$$

where we have dropped the surface index, assuming identical correlations within the monolayers. $g_z(r)$ describes the height correlation between molecules

in the same surface, and $g(r)$ measures the number density correlations in the monolayer, and is in fact the pair correlation function of the two-dimensional liquid. The functions are dimensionless, with the conventional normalization $g(r \rightarrow \infty) = 1$. However, $g_z(r \rightarrow \infty) \rightarrow \infty$ because the height fluctuations grow with the system size.

The integrand in Eq. (S1.7) depends only on the difference $\mathbf{r} - \mathbf{r}'$ which makes it easy to integrate over the primed variable to obtain

$$\langle z_1(\mathbf{q}) z_1^*(\mathbf{q}) \rangle = \frac{\langle z_1^2 \rangle}{N} + \frac{\langle h \rangle^2}{aN} \int_A d^2\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} g_z(r) \quad (\text{S1.11})$$

$$\langle \rho_1(\mathbf{q}) \rho_1^*(\mathbf{q}) \rangle = \frac{1}{a^2 N} + \frac{1}{a^3 N} \int_A d^2\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} g(r) \quad , \quad (\text{S1.12})$$

for the monolayers.

The cross terms are now obtained in an analogous way to Eqs. (S1.7) and (S1.8), yielding

$$\begin{aligned} \langle z_1(\mathbf{q}) z_2^*(\mathbf{q}) \rangle &= \frac{1}{N^2} \iint_{A^2} d^2\mathbf{r} d^2\mathbf{r}' e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} \\ &\quad \times \left\langle \sum_{l,m}^N z_1(\mathbf{r}) z_2(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}_{1l}) \delta(\mathbf{r}' - \mathbf{r}_{2m}) \right\rangle \quad (\text{S1.13}) \end{aligned}$$

$$\begin{aligned} \langle \rho_1(\mathbf{q}) \rho_2^*(\mathbf{q}) \rangle &= \frac{1}{(aN)^2} \iint_{A^2} d^2\mathbf{r} d^2\mathbf{r}' e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} \\ &\quad \times \left\langle \sum_{l,m}^N \delta(\mathbf{r} - \mathbf{r}_{1l}) \delta(\mathbf{r}' - \mathbf{r}_{2m}) \right\rangle \quad , \quad (\text{S1.14}) \end{aligned}$$

with the difference that the $l = m$ terms need not be handled separately since the molecules belong to different surfaces. The correlation functions are

$$\begin{aligned} g_z^c(r = |\mathbf{r}' - \mathbf{r}''|) &= -\frac{a^2}{\langle h \rangle^2} \left\langle \sum_{l,m}^N z_1(\mathbf{r}') z_2(\mathbf{r}'') \right. \\ &\quad \left. \times \delta(\mathbf{r}' - \mathbf{r}_{1l}) \delta(\mathbf{r}'' - \mathbf{r}_{2m}) \right\rangle \quad (\text{S1.15}) \end{aligned}$$

$$g^c(r = |\mathbf{r}' - \mathbf{r}''|) = a^2 \left\langle \sum_{l,m}^N \delta(\mathbf{r}' - \mathbf{r}_{1l}) \delta(\mathbf{r}'' - \mathbf{r}_{2m}) \right\rangle \quad , \quad (\text{S1.16})$$

with the c (“correlation”) superscript to emphasize that the molecules belong to different monolayers. Integration over the primed variable gives

$$\langle z_1(\mathbf{q}) z_2^*(\mathbf{q}) \rangle = -\frac{\langle h \rangle^2}{aN} \int_A d^2\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} g_z^c(r) \quad (\text{S1.17})$$

$$\langle \rho_1(\mathbf{q}) \rho_2^*(\mathbf{q}) \rangle = \frac{1}{a^3N} \int_A d^2\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} g^c(r) \quad . \quad (\text{S1.18})$$

Keeping in mind the factor 2 from the definitions, Eqs. (S1.11), (S1.12), (S1.17) and (S1.18) yield

$$\langle |u(q)|^2 \rangle = \frac{\langle u^2 \rangle + \langle h^2 \rangle}{2N} + \frac{\langle h \rangle^2}{2aN} \int_A d^2\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} (c_z(r) - c_z^c(r)) \quad (\text{S1.19})$$

$$\langle |h(q)|^2 \rangle = \frac{\langle u^2 \rangle + \langle h^2 \rangle}{2N} + \frac{\langle h \rangle^2}{2aN} \int_A d^2\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} (c_z(r) + c_z^c(r)) \quad (\text{S1.20})$$

$$\langle |\rho(q)|^2 \rangle = \frac{1}{2a^2N} + \frac{1}{2a^3N} \int_A d^2\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} (c(r) + c^c(r)) \quad . \quad (\text{S1.21})$$

We have introduced the functions $c(r) = g(r) - 1$ in an obvious way to include the previously ignored δ -functions.

The angular part of the integrals may easily be evaluated since the correlation functions are independent of the angle. With $c_u(r) \equiv c_z(r) - c_z^c(r)$, $c_h(r) \equiv c_z(r) + c_z^c(r)$, and $c_\rho(r) = c(r) - c^c(r)$, this yields

$$\langle |u(q)|^2 \rangle = \frac{\langle u^2 \rangle + \langle h^2 \rangle}{2N} + \frac{\pi \langle h \rangle^2}{aN} \int_{L_0} dr r J_0(qr) c_u(r) \quad (\text{S1.22})$$

$$\langle |h(q)|^2 \rangle = \frac{\langle u^2 \rangle + \langle h^2 \rangle}{2N} + \frac{\pi \langle h \rangle^2}{aN} \int_{L_0} dr r J_0(qr) c_h(r) \quad (\text{S1.23})$$

$$\langle |\rho(q)|^2 \rangle = \frac{1}{2a^2N} + \frac{\pi}{a^3N} \int_{L_0} dr r J_0(qr) c_\rho(r) \quad , \quad (\text{S1.24})$$

where L_0 is the linear system size (radius) and $J_0(x)$ is the zeroth-order Bessel function of the first kind (3). The undulation and number density spectra have already been shown in Fig. 3 in the main text (2). For comparison, the thickness spectrum is shown together with the undulation spectrum in Fig. S1.

S2. Normalization of the Fluctuation Spectra

Consider the structure factors in the $q \rightarrow \infty$ limit. Since $J_0(x \rightarrow \infty) = 0$, the integrals are zero and only the first terms remain. The long-wave

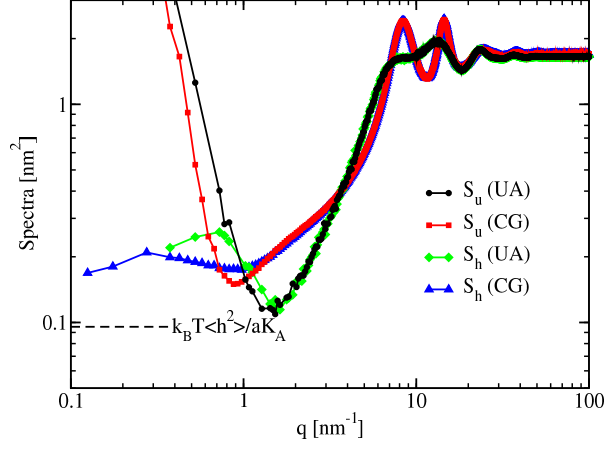


Figure S1: The bilayer undulation and thickness structure factors, $S_u(q)$ and $S_h(q)$, of the UA and CG systems, versus q , together with the theoretical small- q limit.

vector limits of the undulation and thickness structure factors are therefore identical,

$$\langle |u(q \rightarrow \infty)|^2 \rangle = \langle |h(q \rightarrow \infty)|^2 \rangle = \frac{1}{2N} (\langle u^2 \rangle + \langle h^2 \rangle) \quad , \quad (\text{S2.25})$$

while the large- q limit of the number density structure factor is

$$\langle |\rho(q \rightarrow \infty)|^2 \rangle = \frac{1}{2a^2 N} \quad . \quad (\text{S2.26})$$

Therefore, the normalization

$$S_u(q) = N \langle |u(q)|^2 \rangle \quad (\text{S2.27})$$

$$S_h(q) = N \langle |h(q)|^2 \rangle \quad (\text{S2.28})$$

$$S_\rho(q) = Na^2 (\langle u^2 \rangle + \langle h^2 \rangle) \langle |\rho(q)|^2 \rangle \quad , \quad (\text{S2.29})$$

is chosen giving the same large- q limit value, $(\langle u^2 \rangle + \langle h^2 \rangle)/2$ for all spectra.

S3. The Small-Wave Vector Limit of the Number Density and Thickness Spectra

In the $q \rightarrow 0$ limit, the number density spectrum is given by the zero mode of the structure factor, i.e., by

$$S_\rho(q=0) = Na^2 (\langle u^2 \rangle + \langle h^2 \rangle) \langle |\rho(q=0)|^2 \rangle \quad . \quad (\text{S3.30})$$

Inserting $q = 0$ in Eq. (S1.24), and neglecting correlations in number density between the monolayers (i.e., $c^c(r) = 0$), remembering that $J_0(0) = 1$, the structure factor reduces to a pure integral over the correlation function,

$$\langle |\rho(0)|^2 \rangle = \frac{1}{2a^2N} \left[1 + \frac{2\pi}{a} \int_{L_0} dr r c(r) \right] , \quad (\text{S3.31})$$

The term in square brackets is equal to $k_B T / a K_A^m$ by the (two-dimensional) compressibility equation of statistical mechanics [see e.g., Reichl (4)]. Since the bilayer area compressibility is twice that of the monolayer ($K_A = 2K_A^m$), we obtain the relation between K_A and the zero mode of the number density spectrum as

$$S_\rho(0) = \frac{k_B T}{a K_A} [\langle u^2 \rangle + \langle h^2 \rangle] . \quad (\text{S3.32})$$

For a full derivation of the compressibility equation and an ensemble discussion in this context, see Hansen and McDonald (1, Chapter 2) and Forster (5).

Finally, we deduce an approximate relation between the small- q limit of the thickness structure factor and the area fluctuations

$$S_h(0) = N \sigma_h^2 \approx N \left(\frac{\sigma_h}{h} \right)^2 \langle h^2 \rangle \approx N \left(\frac{\sigma_A}{A} \right)^2 \langle h^2 \rangle , \quad (\text{S3.33})$$

where we have replaced h^2 with its ensemble average after the first approximate sign. The second approximation, $\sigma_h/h = \sigma_A/A$ is based on the assumption that thickness and area fluctuations are anti-correlated and thus volume conserving. Using the standard equation relating the area fluctuations to the area compressibility,

$$\left(\frac{\sigma_A}{A} \right)^2 = \left(\frac{\sigma_a}{a} \right)^2 = \frac{k_B T}{N a K_A} , \quad (\text{S3.34})$$

we finally obtain

$$S_h(0) = \frac{\langle h^2 \rangle k_B T}{a K_A} = \frac{\langle h^2 \rangle}{\langle h^2 \rangle + \langle u^2 \rangle} S_\rho(0) . \quad (\text{S3.35})$$

As noted in the main text the factor in front of $S_\rho(0)$ is very close to 1 even for simulations of large systems. From Fig. S1, we note that this equation would fit best with a value of K_A that is about half the experimental value. This may partly be due to the ‘peristaltic’ approximation being a bit inadequate but it is also clear that we might need to go further down in q to reach the small- q limit for this structure factor. In contrast it seems that the density structure factor has reached the small- q limit at these values of q .

Supporting Material References

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