We wish we had supported the use of fractional seats more strongly in Appendix A of the 2021 paper. We had written that district partisan preferences should clearly be interpreted as fractional parts to the two parties under consideration as stated in footnote 19:

"For example, a district that has a 50/50 partisan preference should count as half a seat for both parties. Details for fractional seat assignment and other technical aspects in this paragraph are given in Supplementary Appendix A. Fractional seats have been employed in various ways by Gelman and King (1994, 532) and Cottrell (2019), and the concept is nicely explained by McGann et al. (2016, 58–60) where the 5% variation that is employed in our study was suggested."

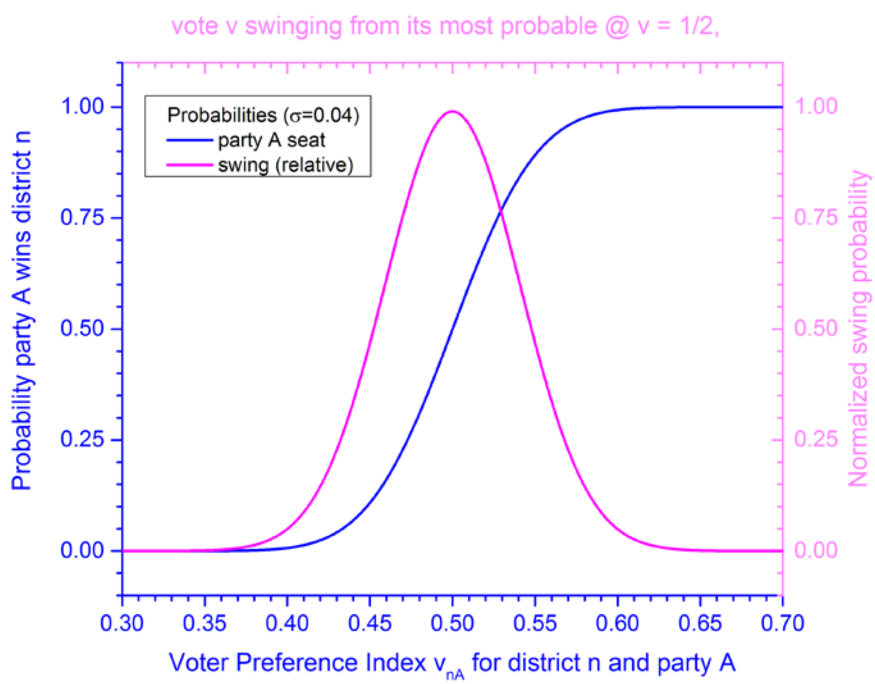
Appendix A continued

"The fraction of a seat estimated for each shifted district was then calculated using party seat probability P(V) = 1 – ½(1+prob((V – ½)/0.04)) where prob is the usual probit function, here with variance 0.04.[2] For example, this function estimates that a district with 55% preference for party A has a seat likelihood of 10.5% for party B as shown in Fig. 1 in Nagle 2019."

This is what we wish we had added:

Fig. 1 is here redrawn with the same P(V) as in the paper with $\sigma = 0.04$. Fig. 1 also shows its derivative $p(v) = (dP(v)/dv)/N$ where N was chosen to normalize $p(0.5) = 1$. This p(v) derivative is the underlying Gaussian $\exp(-(v-\frac{1}{2})^2/2\sigma^2)$ that is the basis of the normal distribution and the probit function and much of the field of statistics.

Fig. 1. This is what is assumed in the paper.



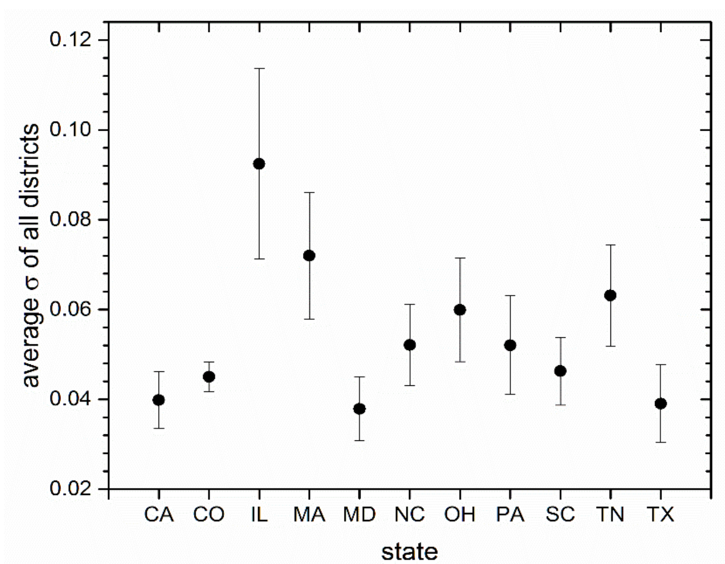
This Gaussian is the probability p(v) that the vote in an election be v votes if the mean value V equals ½. In words, p(v) shows the probability of a swing v – ½ in an election. When the mean vote is 0.5, then the magenta curve shows the probability for swings in the actual vote. Since the sum of the probabilities for v exceeding 0.5 is also 50%, this means that the probability of party A winning is 0.5 and this provides the (½ , ½) point on the blue probability curve. For any other average vote v_{nA} on the blue horizontal axis, the magenta curve is shifted to $v = v_{nA}$ and the sum of all probabilities exceeding ½ is calculated to provide the probability that party A wins the district.

There are two main assumptions here. The first is choosing the probability distribution function to be a Gaussian. The classic argument for this is that Gaussian probability is what ensues from lots of random causes that can't be predicted and that are uncorrelated with each other. This is the standard assumption in statistics and probability since Gauss 200 years ago. However, it is always possible that the distribution follows a different distribution. For example, given an average partisan preference, a district might swing further towards one party less than half the time, but the swing would be larger on average; this is skewness in the distribution of swings. Another example is that the swings are large in both directions with few near zero swings; this is kurtosis in the distribution.

Of the many standard statistical tests for rejecting Gaussian normality, the Shapiro-Wilk p-test was applied using Origin software to the 194 districts in the 11 states in this paper using all the elections in Appendix A. It was found that 25 districts were rejected for normality at the usual 5% level. However, a single outlier election can cause districts to be rejected; this was most obvious in TN where the swing for 2006 Governor was 28% above the mean. Omitting this election reduced the number of TN rejected districts from 2 to 0. The p-test also identified this election as an outlier when it was applied to all the statewide elections. This second use of the p-test was then employed in other states and the 2010 Attorney General election in CA and the 2010 Senate election in SC were also found to be outliers. Removing these elections reduced the number of rejected districts to 5 in CA, 1 in NC and 2 in TX for a total of 8 districts (4.1%) rejected for normality. Of course, 5% of random draws from a Gaussian distribution should be rejected for normality at the 5% level, so the assumption of normality is empirically supported.

The second assumption is the numerical value of the width $\sigma = 0.04$ of the Gaussian distribution. Of course, σ is not expected to be the same for each district or for each state. We can estimate σ from the second moment of the empirical distribution of the swings.[1] The resulting average district σ for each state is shown in the next figure.

Fig. 2. Empirical values of average district σ by state. Average over all states is $\sigma = 0.055 \pm 0.017$. Average weighted by number of districts is $\sigma = 0.051 \pm 0.010$.



[1] This is just the square root of the average squared mean deviation from the average vote.

Choosing different values of σ makes a significant difference in obtaining a fractional seat value for one district. However, when we obtain the sum of fractional seats of a party in a state, the differences in the sum is smaller because there will usually be districts leaning in both directions, although not the same number which is one reason why fractional seats should be calculated.[2] Of course, DRA could be programmed to compute σ for each state and even for each district, although the accuracy will generally be poorer than what is shown in Fig. 2 because there are generally fewer elections in the DRA database than the 9 to 16 in Table A.1.[3]

In summary, using $\sigma = 0$ is all-or-nothing which unrealistically assumes that a district that slightly leans to a party by 1% is as likely to be won by that party as a district that favors that party by 10%.[4] We assert that the use of $\sigma = 0$ is an assumption, and a much worse assumption than assuming normal statistics with a value of σ that is consistent with the literature and strongly supported by data.[5]

[2] In the past results have been compared using $\sigma = 0.02$ with those from using $\sigma = 0.04$ with acceptable differences.

[3] A suitable compromise for DRA would be to calculate σ from the statewide elections. That σ has negligibly smaller values than those shown in Fig.2.

[4] It might be noted that using a very large value of σ gives P(V) = ½ for all V, which unrealistically says that partisan preference makes no difference in future elections.

[5] A criticism of our use of normality tests is that they tend to be more accurate with more than 30 data points, but we only have, depending upon the state, between 9 and 16 elections. A superficially appealing alternative to exceed 30 data points would combine the swings of all districts in a state. However, such a data combination should not be used to test district normality because there is strong correlation in the district swings for each statewide election; analyzing such a data set for normality would require a more complicated analysis with a reduced effective number of independent data points.